CHIRAL CHARGED FERMIONS, ONE DIMENSIONAL QUANTUM FIELD THEORY AND VERTEX ALGEBRAS

Florin Constantinescu

Fachbereich Mathematik, Johann Wolfgang Goethe–Universität Frankfurt,
Robert–Mayer–Strasse 10, D–60054 Frankfurt am Main, Germany

Günter Scharf

Institut für Theoretische Physik, Universität Zürich
Winterthurerstr. 190 CH–8057 Zürich, Switzerland

Abstract

We give an explicit L^2–representation of chiral charged fermions using the Hardy–Lebesgue octant decomposition. In the " pure" case such a representation was already used by M. Sato in holonomic field theory. We study both "pure" and " mixed" cases. In the compact case we rigorously define unsmeared chiral charged fermion operators inside the unit circle. Using chiral fermions we orient our findings towards a functional analytic study of vertex algebras as one dimensional quantum field theory.

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1 Introduction

Let $\mathbf{R}^n = \cup_{\alpha=1}^{2^n}\Gamma_\alpha$ be the octant decomposition of $\mathbf{R}^n$ by octants Γ_α considered with their parity. Each L^2–function $\psi(x)$, $x \in \mathbf{R}^n$ has a decomposition

$$\psi(x) = \sum_{\alpha=1}^{2^n} \psi_\alpha(x + i\Gamma_\alpha 0), i = \sqrt{-1} \tag{1}$$

where ψ_α are L^2–boundary values in tubes $\mathbf{R}^n + i\Gamma_\alpha$ induced by the octant decomposition in the conjugate Fourier variable. Remark that two different boundary values $\psi_\alpha, \varphi_\beta$, $\alpha \neq \beta$ of $\psi, \varphi \in L^2(\mathbf{R}^n)$ are orthogonal, i.e. $\int \psi_\alpha(x)\varphi_\beta(x)\,dx = 0$. We call (1) Hardy–Lebesgue octant decomposition. Certainly, a compact version in which $\mathbf{R}$ is replaced by the unit circle S^1 is also possible by using the Cauchy indicatrix. In this case we have the classical Hardy decomposition. By means of this decomposition we introduce in section 2 chiral charged fermions by an explicit representation in the antisymmetric Fock space $\mathcal{F}(L^2(\mathbf{R}))$. This representation parallels usual canonical anticommutation relations. Charged Fermions polarize the antisymmetric Fock space. Accordingly a graduation of the form

$$\mathcal{F}(L^2(\mathbf{R})) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(L^2(\mathbf{R}) = \bigoplus_{n=0}^{\infty} \bigoplus_{n_1+n_2=n} \mathcal{F}_{n_1,n_2}(L^2(\mathbf{R}) = \bigoplus_{n_1,n_2\geq 0} \mathcal{F}_{n_1,n_2}(L^2(\mathbf{R})) \tag{2}$$

appears where $n_1, n_2 \geq 0$ refer to the octant decomposition and collect all Γ_α with n_1 positive and n_2 negative components. The case $n_1 = n_2 = 0$ corresponds to the vacuum. We call spaces $\mathcal{F}_{n_1,0}$, $\mathcal{F}_{0,n_2}$ ”pure” and others ”mixed”. The ”pure” fermionic sectors were already used by M. Sato and coworkers [1] in holonomic field theory. We insist here on the ”mixed” sectors. In the compact case we rigorously define analytically continued chiral, charged fermionic operators inside the unit circle.

Our findings are analyzed from the point of view of vertex operator algebras realized as one–dimensional quantum field theory. There is no spectral condition in momentum space. Translation invariance and especially locality play the central role. An interesting point is that our quantum fields appear as boundary values of analytic (operator- valued) functions, a property which in the standard Wightman quantum field theory is reserved to vacuum expectation values being a consequence of the spectral condition (and Poincaré invariance). In section 3 in order to establish full contact to vertex algebras we have to pay a price by giving up complex conjugation. This loss is fully compensated by rigorous operator product expansion. Formal algebraic machinery of vertex algebras is enriched by functional analytic framework. The example of chiral charged fermions and other examples worked out in this paper can be extended to more general vertex algebras.

2 Fock space representations of charged fermions

In order to explain what we are going to do let us consider the well–known representation of canonical anticommutation relations in the antisymmetric Fock space $\mathcal{F}(L^2(\mathbf{R}))$:

$$\left(A(f)\psi\right)^n(x_1, \ldots, x_n) = (n + 1)^{1/2} \int dx\overline{f(x)}\psi^{n+1}(x, x_1, \ldots, x_n) \tag{3}$$

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$$\left(A^*(f)\psi\right)^n(x_1,\ldots,x_n) = n^{-1/2}\sum_{i=1}^{n}(-1)^{i-1}f(x_i)$$

$$\times \psi^{n-1}(x_1,\ldots,\hat{x}_i,\ldots,x_n) \tag{4}$$

and in the unsmeared form:

$$\left(A(x)\psi\right)^n(x_1,\ldots,x_n) = (n+1)^{1/2}\psi^{n+1}(x,x_1,\ldots,x_n) \tag{5}$$

$$\left(A^*(x)\psi\right)^n(x_1,\ldots,x_n) = n^{-1/2}\sum_{i=1}^{n}(-1)^{i-1}\delta(x-x_i)$$

$$\times \psi^{n-1}(x_1,\ldots,\hat{x}_i,\ldots x_n) \tag{6}$$

Here $\psi = (\psi^0,\psi^1,\ldots,\psi^n,\ldots)$ where $\psi^n = \psi^n(x_1,x_2,\ldots,x_n) \in L^2(\mathbf{R}^n)$ is an element of the antisymmetric Fock space, ψ^0 is the vacuum and the hat denotes the missing variable. On the vacuum we have

$$A(f)\psi^0 = 0, \quad f \in L^2(\mathbf{R}).$$

The smeared $A(f)$ and $A^*(f)$ are densely defined, closed operators with $A^*(f) = A(f)^*$ being the adjoint of $A(f)$. The unsmeared $A(x)$ is an operator whereas $A^*(x)$ is only a bilinear form. The maps

$$f \mapsto A(f), \quad f \mapsto A^*(f)$$

are antilinear and linear, respectively, and

$$A(f) = \int dx\,\overline{f(x)}A(x), \quad A^*(f) = \int dx\,f(x)A^*(x). \tag{7}$$

The operators $A(f)$ and $A^*(f)$ satisfy the canonical anticommutation relations

$$\{A(f),A(g)\} = \{A^*(f),A^*(g)\} = 0 \tag{8}$$
$$\{A(f),A^*(g)\} = (f,g) \tag{9}$$

where (f,g) is the scalar product in $L^2(\mathbf{R})$. The operators $A(f)$, $A^*(f)$ are bounded with norm

$$\|A(f)\| = \|A^*(f)\| = \|f\|_2 \tag{10}$$

Now we introduce charged fermions $a(f),a^*(f), f \in L^2(\mathbf{R})$ by the following defining relations in antisymmetric Fock space:

$$\left(a(f)\psi\right)^n(x_1,\ldots,x_n) = (n+1)^{1/2}\int \overline{f_+(x)}\psi^{n+1}(x,x_1,\ldots,x_n)\,\mathrm{d}x$$

$$+ n^{-1/2}\sum_{j=1}^{n}(-1)^{j-1}f_-(x_j)\psi^{n-1}(x_1,\ldots,\hat{x}_j,\ldots,x_n) \tag{11}$$

$$(a^*(f)\psi)^n(x_1,\ldots,x_n) = (n+1)^{1/2}\int \overline{f_-(x)}\psi^{n+1}(x,x_1,\ldots,x_n)\,\mathrm{d}x$$

$$+ n^{-1/2}\sum_{j=1}^{n}(-1)^{j-1}f_+(x_j)\psi^{n-1}(x_1,\ldots,\hat{x}_j,\ldots,x_n) \tag{12}$$

where $f = f_+ + f_-$ is the Hardy–Lebesgue decomposition of f.

In compact form the relations (11) and (12) can be given with the help of $A^\#$, ($A^\#$ stays for A or A^*) as

$$a(f) = A(f_+) + A^*(f_-) \tag{13}$$
$$a^*(f) = A(f_-) + A^*(f_+) \tag{14}$$

where $A^*(f_\pm) = A(f_\pm)^*$. Remark the similarity of (13) and (14) to two dimensional (time–independent) Dirac fermions which are also defined in the (antisymmetric) Fock space over the direct sum $H_+ \oplus H_-$. The difference is that for Dirac fermions we start in momentum space and both H_+ and H_- are copies of the same L^2 Hilbert–space. In addition the natural conjugation in H_+ and H_- accounts for the charge conjugation, see for instance [2]. Writing in this case $f = (f_+, f_-)$, $f_\pm \in H_\pm \equiv L^2(\mathbf{R})$ the formal difference is that for Dirac fermions the second term in $a(f)$ has $\overline{f_-}$ instead of f_-, $a^*(f)$ being the adjoint of $a(f)$. In this way no mixed linear/antilinear dependence on f appear, which is typical for (13) and (14), see later. On the other hand it is well known that putting together chiral fermions of opposite chirality one obtains the massless Dirac fermion.

Time–zero Dirac fermion operators satisfy CAR relations whereas chiral charged fermions $a(f), a^*(f)$ defined above satisfy anticommunitation relations of the form

$$\{a(f), a(g)\} = \{a^*(f), a^*(g)\} = 0 \tag{15}$$
$$\{a(f), a^*(g)\} = \langle f, g \rangle \tag{16}$$

where in contradistinction to (9) $\langle f, g \rangle$ is no longer the scalar product in $L^2(\mathbf{R})$ but it is

$$\langle f, g \rangle = \int \overline{f_+(x)} g_+(x) \, \mathrm{d}x + \int \overline{g_-(x)} f_-(x) \, \mathrm{d}x. \tag{17}$$

We call the attention of the reader to the fact that in our case it is not possible to take $\overline{f_-}$ instead of f_- in (13) as in the case of Dirac fermions, which would turn the inner product $\langle \cdot, \cdot \rangle$ into the usual scalar product

$$(f, g) = \int \overline{f_+(x)} g_+(x) \, dx + \int \overline{f_-(x)} g_-(x) \, dx = \int \overline{f(x)} g(x) \, dx,$$

because the relations (15) are no longer true ((16) remains valid). In spite of not being a scalar product, $\langle \cdot, \cdot \rangle$ is positive definite. Indeed

$$\langle f, f \rangle = \int \overline{f_+(x)} f_+(x) + \int \overline{f_-(x)} f_-(x) \, \mathrm{d}x$$

$$= (f, f) = \|f\|_2^2 \tag{18}$$

because $\int \overline{f_+(x)} g_-(x) \, \mathrm{d}x = \int \overline{f_-(x)} g_+(x) \, \mathrm{d}x = 0$ for arbitrary $f, g \in L^2(\mathbf{R})$.

As remarked in the introduction the antisymmetric Fock space decomposes according to

$$\mathcal{F}(L^2(\mathbf{R})) = \bigoplus_{n_1, n_2 \geq 0} \mathcal{F}_{n_1, n_2} = \bigoplus_{n=0}^{\infty} \bigoplus_{n_1 + n_2 = n} \mathcal{F}_{n_1, n_2} = \bigoplus_{n=-\infty}^{\infty} \bigoplus_{n_1 - n_2 = n} \mathcal{F}_{n_1, n_2} \tag{19}$$

giving rise to "pure" and "mixed" states already mentioned. Here n_1, n_2 are the numbers of the a^* and a operators, respectively, applied to the vacuum.

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Neither $a(f)$ nor $a^*(f)$ annihilates the vacuum as in the neutral case. Vacuum expectation values of $a^\#(f)$ satisfy neutrality condition (i.e. they do vanish if the number of a–operators is not equal to the number of a^*–operators) and can be given in closed form. They are Gram determinants in the pure cases and a kind of generalized Gram " determinants" in the mixed case. Moreover they are bounded operators:

$$\|a^*(f)\| = \|f\|_2. \tag{20}$$

It is interesting to remark that the proof of (20) doesn't follow directly from CAR because $\langle \cdot, \cdot \rangle$ in (16) is not a scalar product. Nevertheless the reader can check without difficulty that the usual boundedness proof for fermions [3] can be easily adapted because it only uses positivity of $\langle \cdot, \cdot \rangle$.

The explicit Fock space realization of chiral charged fermions (11) and (12) can be taken over to the compact case S^1.

It is somewhat unpleasant that in the non–compact as well as in the compact case $a(f)$ and $a^*(f)$ do not show sharp linear or anti–linear dependence on f such that the unsmeared $a(z)$, $a^*(z)$, $z \in \mathbf{R}$ or S^1 cannot be looked at as operator-valued "kernels" for $a(f)$, $a^*(f)$. In the compact case this point will be discussed (and improved) in the next section. This is the main point of this paper.

Coming to the end of this section let us remark that the notation $a(f)$ for the chiral fermion is not fortunate. Indeed we want to look at $a(f)$ as a one dimensional field operator and as such a better notation would be $\Phi(f)$. The reason we choose a instead of Φ is explained in the next section where the unsmeared $a(z)$ will be interpreted as a member of a vertex algebra.

3 Towards vertex algebras as one–dimensional quantum field theories

Vertex algebras have been related to the two–dimensional Wightman theory [4]. The results of the preceeding section together with results obtained in [5] show that a natural variant is to start with a one–dimensional quantum field theory. Certainly we cannot expect bona fide Wightman fields but the fields we are suggesting have better behaviours.

Indeed, in Wightman theory, as a consequence of spectral condition together with Poincaré invariance, vacuum expectation values show up as certain boundary values of analytic functions. The fields by themselves do not appear as boundary values.

In contradistinction to that, our one–dimensional fields are boundary values of operator-valued analytic functions. But in order to achieve full contact to vertex algebras we still have to do some work which we accomplish in the frame of our example of chiral fermions. Some other examples follow later. In order to start let us remark that strong tools in quantum field theory and string theory like operator product expansion (OPE) have to be formulated in the unsmeared form, which is from a functional analytic point of view not well defined. In conformal quantum field theory a rigorous way out are vertex algebras in which OPE is formulated in the frame of formal power series, formal Laurent expansions and formal distribution theory.

What we claim in this paper is that the algebraic framework of vertex algebras can be realized in functional analysis and this implies the search for a kernel calculus. We decide to give up complex conjugation and to define all our fields as operators in a given complex domain which will be the unit disc $|z| < 1$. Even in the case of chiral fermions it is not at all clear that this is

possible. The following explicit representation of $a(z)$ and $b(z)$ (instead of $a^*(z)$) as operators inspired from the formal unsmeared version of (11) and (12) solves the problem:

$$\left(a(z)\psi\right)^n(z_1,\ldots,z_n) = (n+1)^{1/2}\psi_+^{n+1}\left(z^{-1},z_1,\ldots,z_n\right)$$

$$+ n^{-1/2}\sum_{j=1}^{n}(-1)^j\frac{1}{z-z_j}\psi^{n-1}(z_1,\ldots,\hat{z}_j,\ldots,z_n) \tag{21}$$

$$\left(b(z)\psi\right)^n(z_1,\ldots,z_n) = (n+1)^{1/2}\psi_-^{n+1}(z,z_1,\ldots,z_n)$$

$$+ n^{-1/2}\sum_{j=1}^{n}(-1)^j\frac{1}{z-z_j^{-1}}\psi^{n-1}(z_1,\ldots,\hat{z}_j,\ldots,z_n), \tag{22}$$

where $z_j \in S^1$, $|z| < 1$. We are now going to show that $a(z), b(z)$ are densely defined operators in fermionic Fock space with a domain of definition presented below. In some sense complex conjugation $z \to \overline{z}$ is simulated in (21) and (22) by the inversion $z \to z^{-1}$. Although these relations are similar to the unsmeared version of (11) and (12), they show particularities not present in (11), (12) which will be made clear by progressing in this section.

We use the notations $\psi_\pm^{n+1}(z,z_1,\ldots,z_n)$ in order to denote Hardy components of ψ^{n+1} in the first variable. For instance for the pure case

$$\psi(z_1,\ldots,z_n) = \det(\varphi^i(z_j))_{i,j=1,\ldots,n}, \quad \varphi^i(z) \in L^2(S^1)$$

we have

$$\psi_\pm(z_1,\ldots,z_n) = \begin{vmatrix} \varphi_\pm^1(z_1)\ldots\varphi_\pm^n(z_1) \\ \varphi^1(z_2)\ldots\varphi^n(z_2) \\ \varphi^1(z_n)\ldots\varphi^n(z_n) \end{vmatrix}.$$

Further $\psi_+^n(z^{-1},\ldots)$ means " wrong boundary value" i.e. we first take $\psi_+^n(z\ldots)$ and then replace z by z^{-1}. In order to make sense of this type of boundary value we will restrict ψ^n to Laurent polynomials instead of Laurent series. This restriction enables us to rigorously define $a(z)$ and $b(z)$ as Fock space operators. Indeed, we identify $\mathcal{F}(L^2(S^1))$ with the corresponding space of analytic functions inside and outside the unit circle obtained by the Hardy decomposition (which are Laurent series) and define $a(z), b(z)$ through (21), (22) on the dense domain in $\mathcal{F}(L^2(S^1))$ obtained by cutting the Laurent series to Laurent polynomials. In order to define products of such operators (beside introducing the normal ordering) we have to extend the definition to some semi–infinite Laurent series. This will be discussed later on in this section.

The reader can check anticommutation relations of a and b, in particular

$$\{a(z), b(w)\} = \delta(z-w) \tag{23}$$

where now

$$\{a(z), b(w)\} = a(z)b(w)|_{|z|>|w|} + b(w)a(z)|_{|w|>|z|} =$$

$$= \frac{1}{z-w}\Big|_{|z|>|w|} + \frac{1}{w-z}\Big|_{|w|>|z|} \tag{24}$$

and $\delta(z-w)$ replaces the formal δ–function [4]:

$$\delta(z-w) = z^{-1}\sum_{n\in\mathbf{Z}}\left(\frac{w}{z}\right)^n = w^{-1}\sum_{n\in\mathbf{Z}}\left(\frac{z}{w}\right)^n = \delta(w-z). \tag{25}$$

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Some hints are given below. This is exactly the state of affair in the vertex algebra frame where by now $a(z)$, $b(z)$ are genuine densely defined operators in Fock space for $|z| < 1$.

Some remarks are in order concerning (23). Let us introduce first some notations similar to those in section 2:

$$a(z) = a_1(z) + a_2(z) \tag{26}$$

$$b(z) = b_1(z) + b_2(z) \tag{27}$$

where

$$\left(a_1(z)\psi\right)^n(z_1,\ldots,z_n) = (n+1)^{1/2}\psi_+^{n+1}\left(z^{-1},z_1,\ldots,z_n\right) \tag{28}$$

$$\left(a_2(z)\psi\right)^n(z_1,\ldots,z_n) = n^{-1/2}\sum_{j=1}^n(-1)^j\frac{1}{z-z_j}$$

$$\times\psi^{n-1}(z_1,\ldots,\hat{z}_j,\ldots,z_n) \tag{29}$$

and similar relations for $b_1(z)$ and $b_2(z)$. The operators $a_i(z), b_i(z)$, $i = 1, 2, |z| < 1$ are defined on Laurent polynomials ψ^n but in order to write down (24) we need products of a_i and b_i. Such products are a priori not defined. Let us discuss the case $a_1(z)b_2(w)$ with $|z| > |w|$. Here we have to extend the domain of definition of $a_1(z)$ from Laurent polynomials to some semi–infinite Laurent series. Indeed, for the first critical term $-\frac{1}{w-z_1^{-1}}$ in $b_2(w)$ we have

$$a_1(z)\left(-\frac{1}{w-z_1^{-1}}\right) = a_1(z)\left(\frac{z_1}{1-wz_1}\right)$$

$$= a_1(z)\left(\sum_{n=0}^\infty w^n z_1^{n+1}\right) = \sum_{n=0}^\infty w^n a_1(z)(z_1^{n+1})$$

$$= \sum_{n=0}^\infty w^n z^{-n-1} = \frac{1}{z-w}$$

for $1 > |z| > |w|$ and this is the correct result. The other term $\frac{1}{w-z}b_1(z)$ with $1 > |w| > |z|$ results from $b_1(w)a_2(z)$ after extending the domain of definition of $b_1(z)$. We have used $|z|, |w| < |z_1|, |z_1|^{-1}$.

The reader is asked to check the usual axioms of vertex algebras for our Fock space representation (21, 22). In particular the translation operator T (which coincides with the Virasoro generator L_{-1}) and the state field correspondence is induced by $a(z), b(z)$ as strongly generating set of fields [4] chapter 4 together with relations

$$a(z)|0>|_{z=0} = -z_1^{-1}, \quad b(z)|0>|_{z=0} = z_1 \tag{30}$$

where $|0>$ is the vacuum and z_1, z_1^{-1} appear as Laurent monomials interpreted as states in our Fock space. Here the vacuum is defined to be the constant Laurent polynomial. Derivatives and Wick products of $a(z), b(z)$ which are defined algebraically in vertex algebras allow straightforward interpretation as Fock space operators. The operators $a(z), b(z)$ defined in (21, 22) also satisfy

$$\{a(z), a(w)\} = 0 = \{b(z), b(w)\} \tag{31}$$

independent of the relative position of z and w inside the unit circle. We give some details.

The proof of (31) is similar to the proof of (23) and is based on the equation

$$\frac{1}{z - z_1}\Big|_+ = 0$$

where $+$ means "wrong boundary value" as defined above. Ordering conditions on z and w are not necessary here as it should be on symmetry reasons. We mention that the relations (28), (29) can be expressed by using the Cauchy kernel in order to generate $\psi_\pm$ from ψ.

The translation operator T can be identified as

$$T\psi^n(z_1, \ldots, z_n) = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \psi^n(z_1, \ldots, z_n) \tag{32}$$

and satisfies [4]

$$[T, Y(a, z)] = \partial Y(a, z) \tag{33}$$

where $Y(a, z)$ is the general element of the vertex algebra generated by $a(z)$ and $b(z)$ and a is the Fock space vector

$$Y(a, z)|0\rangle|_{z=0} = a.$$

In particular, using (30) we have

$$Y(-z_1^{-1}, z) = a(z), \quad Y(z_1, z) = b(z). \tag{34}$$

A more precise definition of $Y(a, z)$ will be given below after introducing the Wick product and the smearing out operation on $a(z)$ and $b(z)$. The Wick product $: a(z)b(z) :$ is defined as

$$: a(z)b(z) := a_2(z)b(z) + b(z)a_1(z) \tag{35}$$

and it is again a densely defined operator for $|z| < 1$. From the explicit relations (21), (22) it is clear why $a_1(z)$ cannot stand in front of $b_2(z)$ (and $b_1(z)$ in front of $a_2(z)$). Formulas giving the Wick product through contour integrals which are well known in conformal quantum field theory and string theory are in our frame rigorous equalities between densely defined operators, instead of being used formally as usual.

Let us remark that the relations (26), (27) suggest the interpretation of $a(z)$ and $b(z)$ as densely defined operator-valued analytic functionals (hyperfunctions). This property persists in all other examples of vertex algebras to follow and is very natural if one remembers that elements of vertex algebras are usually defined as formal Laurent series with operator coefficients. It suggests a one-dimensional Wightman quantum field theory (cf. section 4). In the particular case of chiral charged fermions the densely defined smeared operators $a(f), b(f), f \in L^2(\mathbf{R})$ can be shown to be bounded, as this was the case with chiral fermions in section 2 but we refrain from giving details because this property is incidental here and is not true for other examples to follow. Instead let us discuss the nature of the singularity in $a(z), b(z)$ when z passes from the interior to the boundary of the unit circle. This problem is related to the operator product expansion which was claimed to be under rigorous functional analytic control. To see this the reader can write down for $z = re^{i\theta}, r < 1$

$$a(f) = \lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} a(z) f(e^{i\theta}) e^{i\theta} d\theta \tag{36}$$

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$$b(f) = \lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} b(z)f(e^{i\theta})e^{i\theta}\,d\theta \tag{37}$$

where $a(z), b(z)$ are explicitly given by (21), (22) and $f \in L^2(S^1)$. Formally but suggestive we write instead of (36), (37):

$$a(f) = \frac{1}{2\pi i} \int a(z)f(z)\,dz \tag{38}$$

$$b(f) = \frac{1}{2\pi i} \int b(z)f(z)\,dz \tag{39}$$

with integrals on the unit circle. These operator relations are understood as applied to Laurent polynomials or even on semi-infinite Laurent series according to the definition of the operators $a(z), b(z), |z| < 1$ and the extension of their domain of definition given above. The complex conjugation of test functions which was essential for the linear/anti-linear nature of (11), (12) was here completely ignored. This might spoil some special operator properties (like boundedness) but, as remarked above, this is not the point. What counts is a rigorous natural (linear) interpretation of $a(z), b(z)$ as operator boundary values (operator-valued hyperfunctions). This interpretation is particularly rewarding when one introduces operator product expansions. Indeed the rigorous definition of OPE takes place inside the unit circle and segregates the expected singularity. It is in perfect agreement with formal physical work in string and conformal field theory. As an example we can look at products of the form $a(z)b(w)$ or $b(z)a(w)$ with $|z| > |w|$ which can be analysed exactly as above in the context of verifying (23). This is in contrast to ordinary quantum field theory where even in the free case the corresponding definition looks rather heavy. Indeed in order to formulate a rigorous OPE (and define Wick products) of free fields one first writes down formal expressions and only in a second step gives a smeared out definition over the Fourier transform (see for instance [6]).

Specializing in (38), (39) to monomial test functions we obtain operators satisfying anticommutation relations. They have interesting representations in our Fock space which can be traced back to the reproducing property of the Cauchy kernel. We use them to give a direct definition of $Y(a, z)$ which also works for the general case of vertex algebras by constructing first the Fock space vector a to which $Y(a, z)$ is associated; $a \to Y(a, z)$. Let us start with the formal expression

$$a(z) = \sum_{n \in \mathbf{Z}} a_n z^{-n-1} \tag{40}$$

common in vertex algebras considered as formal Laurent series with operator coefficients. In our setting the coefficients a_n are

$$a_n = a(z^n), \quad n \in \mathbf{Z} \tag{41}$$

where (41) is the particular case of (38) with $f(z) = z^n$. By a similar formula we define the coefficients $b_n, n \in \mathbf{Z}$. It is interesting to remark that both a_n and $b_n, n \in \mathbf{Z}$ are not only densely defined as was the case with $a(z), b(z)$, but in addition the set of Laurent polynomials is an invariant domain of definition with respect to forming products (this was not the case with $a(z), b(w)$; remember the necessity of the argument ordering). Now in order to make a long story short, in the general case we have to consider monomial smearing (41) of the entire generating set of the vertex algebra under consideration [4]. Arbitrary products of such operators with $n < 0$ applied to the vacuum generate the Fock vector a and the correspondence $a \to Y(a, z)$ is given by the formula (4.4.5) of [4] by means of Wick products.

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Let us finally remark that the operator properties of $a(z)$, $b(z)$ are similar to those of vertex operators $V_{\pm 1}(z)$ and their smeared out counterparts obtained in [5], which were introduced by different methods in a different framework. Indeed both $V_{\pm 1}(z)$ exist as densely defined operators for $|z| < 1$ but do not have a meaning for $|z| \geq 1$. As far as the smeared versions of a, b (and $V_{\pm 1}$) is concerned [5] we are allowed to approach the unit circle from the interior. We want to stress that the equations (21, 22) show in a clear way that the smeared operators $a(f)$, $b(f)$, $f \in L^2(S^1)$ involve both f_+ and f_- from the Hardy decomposition $f = f_+ + f_-$ in spite of the fact that $a(z)$ and $b(z)$ are defined only inside the unit circle. This is a central point of our approach. Certainly the similarity between $a(z)$ and $b(z)$ on one side and $V_{+1}(z)$, $V_{-1}(z)$ on the other side is a consequence of the correspondence between bosons and fermions in two dimensions. It appears here in conjunction with [5] at the true operator level.

After the example of chiral fermionic vertex algebra as one-dimensional quantum field theory we proceed to other examples. The simplest one is the $\tilde{u}(1)$ theory generated by currents

$$J(z) = a(z)b(z) \tag{42}$$

where we left out the Wick dots. The precise relations giving $J(z)$ as densely defined operator in Fock space can be assembled from (21) and (22). It is clear that the typical square of the Cauchy kernel makes its appearance. The central statement is the locality of $J(z)$:

$$\{J(z), J(w)\} = \delta_z'(z - w) = -\delta_w'(z - w) \tag{43}$$

and can be verifield in the sense of operators by a direct computation. It also follows by twice applying the Dong lemma (cf. section 4) to $a(z)$, $b(z)$ and $J(z)$. Other examples include non-Abelian generalisation of $\tilde{u}(1)$ like current algebras with currents (see for instance [7])

$$J^a(z) = \sum_{i,j=1}^{N} a_i(z) t_{ij}^a b_j(z) \tag{44}$$

where a_i, b_j are independent chiral fermions and t_{ij}^a are transformation matrices in the defining representation of $su(N)$ such that

$$\mathrm{Tr} t^a t^b = \delta_{ab}$$

$$\sum_a t_{ij}^a t_{kl}^b = \delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}$$

$$[t^a, t^b] = \sum_c f_{abc} t^c$$

$$\sum_{a,b} f_{abc} f_{abd} = 2N\delta_{cd}$$

with f_{abc} being the structure constants. The (mutual) locality of the currents $J^a(z)$ can again be verified by direct computation.

Finally we remark that we can obtain from the charged (complex fermions (21), (22) explicit representations of real fermions which in turn can be used for generating explicit representations of $\widehat{so}(N)$ current algebras at level one or even higher levels (see [7]).

4 Remarks and conclusions

The aim of this paper is twofold. First we gave an explicit representation of chiral charged fermions in Fock space insisting on what we called " mixed states". Second we modified the above (unsmeared) representation in order to get full contact to definitions, methods and techniques in vertex algebras [4]. This is the main point of the paper. Although we have presented only simple examples it is clear that the present results can be generalized. Summarizing we have shown how functional analysis penetrates vertex algebras formulated as one-dimensional quantum field theory.

It would be interesting to develope our findings into a general axiomatic approach to vertex algebras. In the above mentioned framework of one-dimensional quantum field theory one should start with a locally convex algebra of distributional test functions on the circle (Borchers algebra) on which vertex algebra elements are defined as (unsmeared) operators inside the unit disc. If positivity is expected then the factorization and Hilbert space completion common in Wightman theory boil down to a symmetrization property of the Borchers algebra consistent with symmetry properties of the given vacuum expectation values. The Cauchy indicatrix and its variants present in our examples discussed above has to be replaced by some reproducing kernels characterizing the given vertex algebra. This remembers proposals in [8] section 5 and [9] section 8.

We want to stress the very appealing idea of a one-dimensional quantum field theory with fields being operator boundary values (operator-valued hyperfuctions), as opposed to the standard Wightman theory where this property is reserved to vacuum expectation values. This might have drastic consequences. For example we mention the Dong lemma of vertex algebra which in our framework turns out to be a triviality following from the trasitivity of (mutual) locality via weak locality (i.e. commutativity inside expectation values). It is a simple example of the celebrated Borchers transitivity of local quantum field theory.

Another remark concerns the quality of our operators representing elements of vertex algebras. They are densely defined but might be ugly for instance concerning closability (cf. [5]). But nobody would expect more from them as long as they can be multiplied. The examples presented in this paper, i.e. chiral charged fermions and current algebras are easily understood in this framework. In fact, the program can be extended to vertex algebras with a fermionic representation. We do not know how to obtain explicit representations (if any) of the type (21, 22) for general lattice vertex algebras, although our feeling is that such representations could exist.

Last but not least an explicit operator realization of the type (21), (22) and its smeared counterpart can be used to study the C^*–content of vertex algebras. This seems to be usefull in the context of recent developement at the interface between strings and non–commutative geometry; see for instance [10].

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